Eur. Phys. J. C 21, 735–747 (2001)
Digital Object Identifier (DOI) 10.1007/s100520100769

THE EUROPEAN PHYSICAL JOURNAL C

Non-trivial extension of the $(1+2)$-Poincaré algebra and conformal invariance on the boundary of AdS_3

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Received: 20 July 2000 /
Published online: 19 September 2001 –

Abstract. Using recent results on strings on $\mathrm{AdS}_3 \times N^d$, where N is a d dimensional compact manifold, we re-examine the derivation of the non-trivial extension of the $(1+2)$-dimensional-Poincaré algebra obtained by Rausch de Traubenberg and Slupinsky. We show by explicit computation that this new extension is a special kind of fractional supersymmetric algebra which may be derived from the deformation of the conformal structure living on the boundary of AdS_3. The two $so(1,2)$ Lorentz modules of spin $\pm 1/k$ used in building of the generalization of the $(1+2)$ Poincaré algebra are re-interpreted in our analysis as highest weight representations of the left and right Virasoro symmetries on the boundary of AdS_3. We also complete known results on 2d-fractional supersymmetry by using spectral flow of affine Kac–Moody and superconformal symmetries. Finally we make preliminary comments on the trick of introducing Fth roots of g-modules to generalize the $so(1,2)$ result to higher rank Lie algebras g.

1 Introduction

Recently a non-trivial generalization of the $(1+2)$-dimensional Poincaré algebra going beyond the standard supersymmetric extension has been obtained in [1]. In addition to the usual Poincaré generators, this extension, here referred to as the Rausch de Traubenberg–Slupinski algebra (RdTS algebra for short), involves two kinds of conserved charges $Q_s^{\pm}$ transforming as $so(1,2)$ Verma modules of spin $s = \pm 1/k; k \geq 2$. This construction is interesting first because it goes beyond standard 2d-fractional supersymmetry based on considering kth roots of the $so(2)$ vector, and second because it gives a new algebraic structure which a priori is valid for higher rank Lie algebras g where $so(2)$ and $so(1,2)$ appear just as two special examples. In two dimensions where conformal invariance is infinite we now know, by the help of conformal field theory methods and techniques of complex analysis, how to deal with objects of the type of the kth root of a $so(2)$ vector. For higher space-time dimensions however, computations are in general difficult to perform except for some special situations such as the problem we will study low and where RdTS symmetry finds applications in low dimensional physical systems.

In $(1+2)$ dimensions, representations of the RdTS extension of the $so(1,2)$ algebra have quantum states carrying fractional values of the spin and are expected to play a particular role in the exploration of special features of field theoretical models of $(1+2)$-dimensional systems with boundaries. The idea of considering 3d systems with boundaries is crucial. It is motivated by the fact that one can imagine that the RdTS $so(1,2)$ extension may naturally be linked to a 2d boundary conformal field theory (BCFT) living on the boundary of the space-time. From this view we expect that the RdTS construction for $so(1,2)$ may be related to known results on integrable deformations of 2d conformal invariance. Recall that representations theory of conformal invariance in two dimensions [2] predict naturally the existence of quantum field operators generating states with exotic spins englobing the $so(1,2)$ RdTS ones. It is then an interesting task to check if there exists effectively any relation between the RdTS generalization of Poincaré invariance in $(1+2)$ dimensions and known results on integrable deformations of 2d CFT's [3,4]. We expect that this relation really exists, and its determination may help in understanding the behavior of physical bulk quantities near the boundary of $(1+2)$-dimensional systems.

To study this problem we shall mainly work with AdS_3 as the $(1+2)$ space-time with boundary and use recent results on strings propagating on $\mathrm{AdS}_3 \times N^d$, where N^d is a d-dimensional compact manifold to be specified later on. The analysis we will develop in this paper might also be adapted to study some features of fractional quantum Hall (FQH) effects [5,6]; in particular the understanding of the correspondence between the bulk effective Chern–Simons (CS) gauge theory of FQH droplets and the conformal field theory living on its boundary [6,7].

The aim of this paper is to exhibit explicitly the link between the RdTS analysis and 2d BCFT using recent results on D-brane physics on the (1+2)-dimensional anti-de Sitter space AdS_3 [8–10]. We first show that there exists indeed a connection between the RdTS algebra and deformations of 2d space-time BCFT. Then we establish the rule of correspondance between the two $so(1,2)$ Verma modules, used in constructing the non-trivial extension of the $(1+2)$ Poincaré invariance, and primary Virasoro representations of the full conformal algebra on the boundary of AdS_3. We show moreover that the RdTS supersymmetry, although obtained using an unusual method, has in fact the same origin as standard fractional supersymmetry (FSS) [11–13]; see also [14,15]. Both FSS and RdTS algebras are residual subsymmetries of conformal invariance.

The presentation of this paper is as follows: In Sect. 2, we review the basic ideas of FSS and RdTS supersymmetry using the conformal field theoretical method for the first and the algebraic approach for the second. We give explicit calculations for the deformation of the $C = 4/5$ Potts model. In Sect. 3, we review the main lines of RdTS analysis. We also introduce some useful tools for the study of the link between the RdTS modules and highest weight representations (HWR) of the Virasoro algebra. In Sect. 4, we study the relation between RdTS supersymmetry and 2-dimensional conformal invariance. We show in particular that the two $so(1,2)$ modules considered in building supersymmetry are just special HWRs of the conformal invariance on the boundary of AdS_3. In Sect. 5, we use the spectral flow of 2d $N = 2$ and $N = 4$ superconformal invariances to complete the study of Sect. 2 by giving a new result on FSS. We also take the opportunity of using spectral flow of affine Kac–Moody symmetries to give comments on the kth roots of the $su(n)$ fundamental representations used by RdTS in extending their result for $so(1,2)$ for $su(n)$. In Sects. 6 and 7 we give our results and conclusions.

2 Fractional supersymmetry

RdTS fractional supersymmetry is a special generalization of FSS living in two dimensions and considered in many occasions in the past in connection with integrable deformations of conformal invariance and representations of the universal enveloping $U_q sl(2)$ quantum ordinary and affine symmetries [11,12,16,17]. Like for FSS, highest weight representations of the RdTS algebra carry fractional values of the spin and obey more or less quite similar FSS equations. We will show throughout this study that, up to some details related to the number of dimensions of space-time, RdTS fractional supersymmetry has indeed the same origin as FSS. Both FSS and RdTS invariance describe residual symmetries left after integrable deformations of scale invariance in two dimensions. To better understand the algebraic structure of FSS and RdTS supersymmetry we first propose to describe briefly the main lines of 2d FSS one gets from integrable deformations of conformal invariance. Then we give the RdTS extension of the $(1+2)$-dimensional Poincaré invariance as derived in [1].

2.1 2-dimensional FSS

FSS extends the usual Bose–Fermi symmetry in two dimensions; it exchanges bosons and quasiparticles (parafermions) of fractional spin instead of fermions. In addition to the energy momentum translation operator vector $P_\pm$, FSS is generated by conserved charges Q_x and $\bar{Q}_x$ carrying fractional values of the spin x ($x = l/k; 1 < l < k \bmod[1]; k \geq 2$). These charge operators are remnant constants of motion that survive after integrable deformations of conformal invariance. There are various FSS algebras depending on the conformal model one starts with. For example, via the Z_k parafermionic invariance of Zamolodchikov and Fateev (ZF) [18]; see also [19], a way to get FSS algebras is as follows. First start from the ZF conformal algebra generated by the energy momentum tensor $T(z)$ and the parafermionic currents $\Psi_q(z), q = 1, \dots k$:

$$\begin{aligned}
T_\Psi(z_1)T_\Psi(z_2) &= c_\Psi/2 z_{12}^{-4} + 2z_{12}^{-2}T(z_2) \\
&\quad + z_{12}^{-1}\partial T(z_2) + \dots, \\
\Psi_k(z_1)\Psi_l(z_2) &= C_{k,l}^{k+l} z_{12}^{-2kl/N}\{\Psi_{k+l}(z_2) + \dots\}, \\
&\quad (k+l) < N, \\
\Psi_k(z_1)\Psi_k^+(z_2) &= C_{k,N-l}^{N+k-l} z_{12}^{-2k(N-l)/N}\{\Psi_{k-l}(z_2) + \dots\}, \\
\Psi_k(z_1)\Psi_k^+(z_2) &= z_{12}^{-2k(N-k)/N} \\
&\quad \times[1_{id} + 2\Delta_k/c_k z_{12}^2 T_\Psi(z_2) + \dots], \\
T_\Psi(z_1)\Psi_k(z_2) &= \frac{\Delta_k}{z_{12}^2}\Psi_k(z_2) + \frac{1}{z_{12}}\partial_z\Psi_k(z_2) + \dots, \qquad (2.1)
\end{aligned}$$

where the parameters c_Ψ and $C_{k,l}^{k+l}$ are the central charges and structure constants of the parafermionic algebra respectively. The $\Psi_q(z)$'s and the $\bar{\Psi}_q(\bar{z})$ have the conformal weights $\Delta_q = q(k-q)/k$. Second, solve the following operator equations:

$$\begin{aligned}
P_- &= \oint \mathrm{d}z T(z), \\
P_+ &= \oint \mathrm{d}\bar{z}\bar{T}(\bar{z}), \qquad (2.2)
\end{aligned}$$

where $T(z)$ and $\bar{T}(\bar{z})$ are replaced by their expressions in terms of the $\Psi^\pm(z)$'s and the $\bar{\Psi}^\pm(\bar{z})$; see (1). To solve these equations, one has to specify the ZF parafermionic primary representations since the mode expansions of the Ψ_q's and the $\bar{\Psi}_q$'s depend on the weight of the ZF primary field operators Φ_q^p.

$$\begin{aligned}
\Psi_k(z_1)\Phi_p^q(z_2) &= \sum_{n\in Z} z_{12}^{n-kp/N-k} Q_{-n+\frac{k(p+k)}{N}}^{k,p}\Phi_p^q(z_2), \\
\Psi_k^+(z_1)\Phi_p^q(z_2) &= \sum_{n\in Z} z_{12}^{n+kp/N-k} Q_{-n-\frac{k(p+k)}{N}}^{-k,p}\Phi_p^q(z_2), \qquad (2.3)
\end{aligned}$$

where $Q_{-n+(k(p+k))/N}^{k,p}$ and $Q_{-n-(k(p+k))/N}^{-k,p}$ are the modes of Ψ_k and Ψ_k^+ respectively, defined by

$$\begin{aligned}
Q_{-n+\frac{k(p+k)}{N}}^{k,p}\Phi_p^q(z_2) &= \oint \mathrm{d}z_1 z_{12}^{n+kp/N+k-1}\Psi(z_1)\Phi_p^q(z_2), \\
Q_{-n-\frac{k(p+k)}{N}}^{-k,p}\Phi_p^q(z_2) &= \oint \mathrm{d}z_1 z_{12}^{n-kp/N+k-1}\Psi(z_1)\Phi_p^q(z_2). \\
&\qquad (2.4)
\end{aligned}$$

To illustrate how things work in practice let us consider an example. The method we will present below applies to all Z_k parafermionic models as well as others such as the symmetries due to Tye et al. [20,21].

2.2 Deformation of $C = 4/5$ Potts model

To fix the ideas, we consider the $c = 4/5$ critical Potts model described by the following Z_3 parafermionic invariance. This is the leading non-trivial example having constants of motion carrying fractional values of the spin. The algebra governing the critical behavior of this model is:

$$\begin{aligned} \Psi^{\pm}(z_1)\Psi^{\pm}(z_2) &\approx -z_{12}^{-2/3}\Psi^{\pm}(z_2), \\ \Psi^{+}(z_1)\Psi^{-}(z_2) &\approx z_{12}^{-4/3}[1+5/3z_{12}^2T(z_2)], \\ T(z_1)\Psi^{\pm}(z_2) &\approx \frac{2/3}{z_{12}^2}\Psi^{\pm}(z_2)+\frac{1}{z_{12}}\partial_z\Psi^{\pm}(z_2), \\ T(z_1)T(z_2) &= 2/5z_{12}^{-4}+2z_{12}^{-2}T(z_2) \\ &\quad +z_{12}^{-1}\partial T(z_2). \end{aligned} \tag{2.5}$$

Similar relations are valid for the $\bar{\Psi}^{\pm}(\bar{z})$'s. The ZF parafermionic currents $\Psi^{\pm}$ have a spin $2/3$ and satisfy $([\Psi^{\pm}(z)]^{+} = \Psi^{\mp}(z))$.

The algebra (2.4) has three parafermionic highest weight representations (PHWR)$[\Phi_q^q]$; $q = 0,1,2$, namely the identity family $I = [\Phi_0^0]$ of highest weight $h_0 = 0$ and two degenerate families $[\Phi_1^1]$ and $[\Phi_2^2]$ of weights $h_1 = h_2 = 1/15$. Each one of these PHWRs is reducible into three Virasoro HWRs: $(\Phi_q^p); p = q, p = q\pm 2$ (mod6). These field operators which are rotated amongst others under the action of the parafermionic currents as shown here:

$$\begin{aligned} \Psi^{\mp}\times\Phi_q^p &= \Phi_q^{p\pm 2}, \\ \Phi_q^{p\pm 6} &= \Phi_q^p, \end{aligned} \tag{2.6}$$

obey Virasoro and ZF primary conditions:

$$\begin{aligned} L_n|h\rangle &= 0, \quad n>0, \\ Q^{\pm}_{-n\pm(p\pm 1)/3}|h\rangle &= 0, \quad n\pm(p\pm 1)/3>0, \end{aligned} \tag{2.7}$$

where the L_n Virasoro and the $Q^{\pm}_{-n\pm(p\pm 1)/3}$ ZF modes are given by

$$\begin{aligned} L_n|\Phi_p^q\rangle &= \oint \mathrm{d}z z^{n+1}T(z)\Phi_p^q(0)|0\rangle, \\ Q^{\pm}_{-n\pm(p\pm 1)/3}|\Phi_p^q\rangle &= \oint \mathrm{d}z z^{n\pm p/3}\Psi^{\pm}(z)\Phi_p^q(0)|0\rangle. \end{aligned} \tag{2.8}$$

Note that the mode expansion of the ZF currents depend on the representation field operator on which they act. This property is manifestly seen on the energies of the creation and annihilation operators $Q^{\pm}_{-n\pm(p\pm 1)/3}$ which depend on the quantum number p of the ZF primary field $\Phi_q^p(z)$:

$$\Psi^{\pm}(z_1)\Phi_q^p(z_2) = \sum z_{12}^{n-1\mp p/3}Q^{\pm}_{-n\pm(p\pm 1)/3}\Phi_q^p(z_2), \tag{2.9}$$

The ZF primary field operators $\Phi_q^p(z)$ satisfy also braiding properties of type

$$z_{12}^{\Delta}\Phi_1(z_1)\Phi_2(z_2) = z_{21}^{\Delta}\Phi_2(z_2)\Phi_1(z_1), \tag{2.10}$$

where $\Delta = \Delta_1+\Delta_2-\Delta_3$; Δ_1 and Δ_2 are respectively the conformal weights of the Φ_1 and Φ_2 field operators while Δ_3 is the weight of Φ_3 fields operators arising when computing the OPE, (2.10).

The second step in the derivation of FSS is to solve the operator (2.2) expressing the 2d energy momentum vector $P_{\pm}$ in terms of the ZF modes $Q^{\pm}_{-n\pm(p\pm 1)/3}$:

$$\begin{aligned} P_- &= \oint \mathrm{d}z\frac{3}{5}z^{-2/3}(\Psi^{+}(z)\Psi^{-}(0)), \\ P_+ &= \oint \mathrm{d}\bar{z}\frac{3}{5}z^{-2/3}(\bar{\Psi}^{+}(\bar{z})\bar{\Psi}^{-}(0)), \end{aligned} \tag{2.11}$$

where we expressed $T(z)$ and $\bar{T}(\bar{z})$ in terms of the $\Psi^{\pm}(z)$'s and the $\bar{\Psi}^{\pm}(\bar{z})$ as given by (2.4). The solution of (2.9) involves three pairs of doublets of the charge operators $\left(Q^{\pm}_{-1/(3)},\bar{Q}^{\pm}_{1/3}\right)$, $\left(Q^{\pm}_{-2/3},\bar{Q}^{\pm}_{2/3}\right)$ and $(Q_0{}^{\pm},\bar{Q}_0^{\pm})$. Using the primary highest weight conditions (2.7), one can check by explicit computation that the $Q,\bar{Q},P_-$ and P_+ charge operators generate the following algebra:

$$\begin{aligned} \mathbf{P} &= Q^{+}_{-1/3}Q^{+}_0Q^{+}_{-2/3}\Pi_0+Q^{+}_{-2/3}Q^{+}_{-1/3}Q^{+}_0\Pi_1 \\ &\quad +Q^{+}_0Q^{+}_{-2/3}Q^{+}_{-1/3}\Pi_{-1}, \\ [P_{\pm},Q_{-x}] &= 0; \quad x = 0,1/3,2/3, \\ \bar{P} &= \bar{Q}^{+}_{-1/3}\bar{Q}^{+}_0\bar{Q}^{+}_{-2/3}\bar{\Pi}_0+\bar{Q}^{+}_{-2/3}\bar{Q}^{+}_{-1/3}\bar{Q}^{+}_0\bar{\Pi}_1 \\ &\quad +\bar{Q}^{+}_0\bar{Q}^{+}_{-2/3}\bar{Q}^{+}_{-1/3}\bar{\Pi}_{-1}, \\ [P_{\pm},\bar{Q}_{+x}] &= 0. \end{aligned} \tag{2.12}$$

In these equations the Π_q's and $\bar{\Pi}_q$'s are projector operators on the qth ZF primary state $[\Phi_q{}^q\times\bar{\Phi}_q^q]$. The algebra (2.12) may also be obtained by analysing the energy spectrum of the mode operators $Q^{\pm}_{-n\pm(p\pm 1)/3}$ and $\bar{Q}^{\pm}_{-n\pm(p\pm 1)/3}$, n integer. The $Q^{\pm}_{-n\pm(p\pm 1)/3}$'s and $\bar{Q}^{\pm}_{-n\pm(p\pm 1)/3}$'s, which depend on the p charge, act only on the conformal representation $|\Phi_p^q\rangle$. This property may be interpreted to mean that, as expected for the $|\Phi_p^q\rangle$ family, the action of the $Q^{\pm}_{-n\pm(p\pm 1)/3}$'s kills all states $|\Phi_p{}^r\rangle$ with r different from q. For $q = 0$ for example, the non-vanishing actions of $Q^{\pm}_{-x}$ and $\bar{Q}^{\pm}_{-x}$, $x = 0,1/3,2/3$ on the states $|s,p\rangle$ of spin s, $0\leq s\leq 1$, and charge p read

$$\begin{aligned} Q^{\pm}_{-2/3}|0,0\rangle &= |2/3,0\rangle, \\ Q^{+}_0|2/3,+2\rangle &= |2/3,-2\rangle, \\ Q^{-}_0|2/3,-2\rangle &= |2/3,+2\rangle, \end{aligned} \tag{2.13}$$

$$\begin{aligned} Q^{+}_{-1/3}|2/3,-2\rangle &= |1,0\rangle, \\ Q^{-}_{-1/3}|2/3,+2\rangle &= |1,0\rangle, \end{aligned} \tag{2.14}$$

and similar equations for the antiholomorphic sector. From these equations as well as the expansion (2.3) and

(2.4) of the ZF currents, one sees that $Q^{\pm}_{-1/3}$ and $Q^{\pm}_{0}$ cannot act directly on the state $|0,0\rangle$. Similarly $Q^{\pm}_{-2/3}$ cannot operate directly on $|2/3,\pm 2\rangle$. This result gives an explicit argument showing that FSS should be generated by more than one Q and $\bar{Q}$ operators as it was naively used in earlier physical literature on FSS. It shows moreover that not all the $Q^{\pm}_{-x}$'s are independent since we have

$$\begin{aligned} Q^{-}_{-1/3} &= Q^{+}_{-1/3} Q^{+}_{0}, \\ Q^{+}_{-1/3} &= Q^{-}_{0} Q^{-}_{-1/3}, \\ Q^{-}_{-2/3} &= Q^{+}_{0} Q^{-}_{-2/3}, \\ Q^{+}_{-2/3} &= Q^{-}_{-2/3} Q^{-}_{0}. \end{aligned} \tag{2.15}$$

Similar expressions may be written down for the antiholomorphic sector. Putting back these relations into (2.12), we find the following linearized algebra:

$$\begin{aligned} 2P_{-1} &= \{Q^{+}_{-2/3}, Q^{-}_{-1/3}\} + \{Q^{+}_{-1/3}, Q^{-}_{-2/3}\}, \\ 0 &= \{Q^{\pm}_{-1/3}, Q^{\pm}_{-1/3}\} = \{Q^{\pm}_{-2/3}, Q^{\pm}_{-2/3}\}. \end{aligned} \tag{2.16}$$

We shall return to this linearized realization of FSS in Sect. 5 when we discuss the spectral flow of $N = 2$ and $N = 4$ superconformal invariance in two dimensions, where a similar result will be obtained by using special choices of the parameter of the flow.

3 RdTS supersymmetry

In this section we review briefly the derivation of the RdTS extension of the $(1 + 2)$-dimensional Poincaré invariance. We also initiate the study of a field realization of RdTS supersymmetry which we develop further in the next section. In this regard we would like to note that as far as the $SO(1,2)$ group is concerned, we will encounter in our analysis various kinds of $SO(1,2)$ symmetries with different physical interpretations. In addition to the $SO(1,2)$ Lorentz invariance of the $(1 + 2)$-dimensional space-time considered in [1], we will handle four $SO(1,2)$ invariances which can be classified:

(1) Two $SO(1,2)$'s given by the zero mode subgroup product $SO(1,2) \times \bar{SO}(1,2)$ associated to $so_k(1,2) \times \bar{so_k}(1,2)$ affine Kac–Moody invariance to be studied in Sect. 4. This subsymmetry will be realized by using the usual $Sl(2,R) \sim SO(1,2)$ Wakimoto field theoretical representation [22].
(2) Two other $SO(1,2)$ subsymmetries associated to the non-anomalous subalgebras of the left and right Virasoro symmetries of some 2-dimensional BCFT of AdS_3 to be specified later on.

To start, consider the Poincaré symmetry in (1+2) dimensions generated by the space-time translations P_μ and the Lorentz rotations J_α satisfying altogether the following closed commutation relations:

$$\begin{aligned} [J_\alpha, P_\beta] &= \mathrm{i}\epsilon_{\alpha\beta\gamma}\eta^{\gamma\delta}P_\delta, \\ [J_\alpha, J_\beta] &= \mathrm{i}\epsilon_{\alpha\beta\gamma}\eta^{\gamma\delta}J_\delta, \\ [P_\mu, P_\nu] &= 0. \end{aligned} \tag{3.1}$$

In these equations, $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1)$ is the $(1 + 2)$ Minkowski metric and $\epsilon_{\alpha\beta\gamma}$ is the completely antisymmetric Levi-Civita tensor such that $\epsilon_{012} = 1$. A convenient way to handle (3.1) is to work with an equivalent formulation using the following Cartan basis of generators $P_\mp = P_1 \pm \mathrm{i}P_2$ and $J_\mp = J_1 \pm \mathrm{i}J_2$. In this basis (3.1) read

$$\begin{aligned} [J_+, J_-] &= -2J_0, \\ [J_0, J_\pm] &= \pm J_\pm, \\ [J_\pm, P_\mp] &= \pm P_0, \\ [J_+, P_+] &= [J_-, P_-] = 0, \\ [J_0, P_0] &= [P_\pm, P_\mp] = 0. \end{aligned} \tag{3.2}$$

The algebra (3.1) and (3.2) has two Casimir operators, $P^2 = {P_0}^2 - 1/2(P_+P_- + P_-P_+)$ and $P.J = P_0J_0 - 1/2$ $(P_+J_- + P_-J_+)$. When acting on highest weight states of mass m and spin s, the eigenvalues of these operators are m^2 and ms respectively. For a given s, one distinguishes two classes of irreducible representations: massive and massless representations. To build the $so(1,2)$ massive representations, it is convenient to go to the rest frame where the momentum vector P_μ is $(m,0,0)$ and the $SO(1,2)$ group reduces to its abelian $SO(2)$ little subgroup generated by J_0 ($J_\pm = 0$). In this case, massive irreducible representations are 1-dimensional and are parameterized by a real parameter. For the full $SO(1,2)$ group however, the representations are either finite dimensional for $|s| \in \mathbf{Z}^+/2$ or infinite dimensional otherwise.

Given a primary state $|s\rangle$ of spin s, and using the above mentioned $SO(1,2)$ group theoretical properties, one may construct in general two representations HWR(I) and HWR(II) out of this state $|s\rangle$. The first representation HWR(I) is a highest weight representation given by

$$\begin{aligned} J^0|s\rangle &= s|s\rangle, \\ J_-|s\rangle &= 0, \\ |s,n\rangle &= \sqrt{\frac{\Gamma(2s)}{\Gamma(2s+n)\Gamma(n+1)}}(J_+)^n|s\rangle, \quad n\geq 1, \\ J_0|s,n\rangle &= (s+n)|s,n\rangle, \\ J_+|s,n\rangle &= \sqrt{(2s+n)(n+1)}|s,n+1\rangle, \\ J_-|s,n\rangle &= \sqrt{(2s+n-1)n}|s,n-1\rangle. \end{aligned} \tag{3.3}$$

The second representation, to which we refer as HWR(II), is a lowest weight representation defined by

$$\begin{aligned} \bar{J}_0|\bar{s}\rangle &= -s|\bar{s}\rangle, \\ \bar{J}_+|\bar{s}\rangle &= 0, \\ |\bar{s},n\rangle &= (-)^n\sqrt{\frac{\Gamma(2s)}{\Gamma(2s+n)\Gamma(n+1)}}(\bar{J}_-)^n|\bar{s}\rangle, \\ \bar{J}_0|\bar{s},n\rangle &= -(s+n)|\bar{s},n\rangle, \\ \bar{J}_+|\bar{s},n\rangle &= -\sqrt{(2s+n-1)n}|\bar{s},n+1\rangle. \end{aligned} \tag{3.4}$$

Note that the generators $\bar{J}_{0,\pm}$ and the representations states $|\bar{s}\rangle$ of the second module carry a bar index. This is a conventional notation which will be justified later on. To fix the ideas, HWR(I) will be identified in Sects. 6 and 7 with a left Virasoro Verma module and HWR(II) will be interpret as a right Virasoro one. Note also that both HWR(I) and HWR(II) representation have the same $so(1,2)$ Casimir $C_s = s(s-1), s<0$. For $s \in \mathbf{Z}^-/2$, these representations are finite dimensional and their dimension is $(2|s|+1)$. For generic real values of s, the dimension of the representations is however infinite. If one chooses a fractional value of s, say $s=-1/k$, each of the two representations (3.3) and (3.4) splits a priori into two isomorphic representations respectively denoted as $D^+_{\pm 1/k}$ and $D^-_{\pm 1/k}$. This degeneracy is due to the redundancy in choosing the spin structure of $(-2/k)^{1/2}$ which can be taken either as $+\mathrm{i}(2/k)^{1/2}$ or $-\mathrm{i}(2/k)^{1/2}$. These representations are not independent since they are related by conjugations; this is why we shall use hereafter the choice of [1] by considering only $D^+_{-1/k}$ and $D^-_{-1/k}$. In this case the two representation generators $J_{0,\pm}$ and $\bar{J}_{0,\pm}$ are related by

$$\bar{J}_{0,\mp} = (J_{0,\pm})^*. \tag{3.5}$$

Furthermore, taking the tensor product of the primary states $|s\rangle$ and $|\bar{s}\rangle$ of the two $so(1,2)$ modules HWR(I) and HWR(II) and using (3.3) and (3.4), it is straightforward to check that it behaves like a scalar under the full charge operator $J_0 \times 1_d + 1_d \times \bar{J}_0$ which we denote simply as $J_0 + \bar{J}_0$:

$$(J_0 + \bar{J}_0)|s\rangle \otimes |\bar{s}\rangle = 0. \tag{3.6}$$

Equation (3.6) is a familiar relation in the study of primary states of the Virasoro algebra. This equation together with the mode operators J^n_- and $\bar{J}^m_+$ which act on $|s\rangle \otimes |\bar{s}\rangle$ by

$$\begin{aligned}(J_-)^n|s\rangle \otimes |\bar{s}\rangle &= 0, \quad n\geq 1,\\ (\bar{J}_+)^m|s\rangle \otimes |\bar{s}\rangle &= 0, \quad m\geq 1,\end{aligned} \tag{3.7}$$

define a highest weight state which looks like a Virasoro primary state of spin $2s$ and scale dimension $\Delta = 0$. We will show later on when we study the primary field representation of the 2d BCFT of a string propagating on the AdS_3 background, that (3.6) and (3.7) indeed correspond to

$$\begin{aligned}(L_0 - \bar{L}_0)\Phi_{h,\bar{h}}(0,0)|0\rangle &= (h-\bar{h})\Phi_{h,\bar{h}}(0,0)|0\rangle,\\ (L_0 + \bar{L}_0)\Phi_{h,\bar{h}}(0,0)|0\rangle &= (h+\bar{h})\Phi_{h,\bar{h}}(0,0)|0\rangle,\\ L_n\Phi_{h,\bar{h}}(0,0)|0\rangle &= 0, \quad n\geq 1,\\ \bar{L}_m\Phi_{h,\bar{h}}(0,0)|0\rangle &= 0, \quad m\geq 1,\end{aligned} \tag{3.8}$$

where L_n and $\bar{L}_m$ are respectively the usual left and right Virasoro modes and $\phi_{h,\bar{h}}(z,\bar{z})$ is a primary conformal field representation of conformal scale $h+\bar{h}$ and conformal spin $h-\bar{h}$. This property, which gives the connection between RdTS supersymmetry and conformal invariance, will be made explicit in detail when we discuss HWRs of the conformal symmetry on the boundary of AdS_3. The primary $so(1,2)$ highest weight states $|s\rangle$ and $|\bar{s}\rangle$ (3.3) and (3.4) are respectively in one to one correspondence with the left Virasoro primary state $\Phi_h(0)|0\rangle = |h\rangle$ and the right Virasoro primary one $\Phi_{\bar{h}}(0)|0\rangle = |\bar{h}\rangle$.

On the other hand, if we respectively associate to HWR(I) and HWR(II) the mode operators $Q^+_{s+n} = Q_{s+n}$ and $Q^-_{-s-n} = \bar{Q}_{s+n}$ and using $SO(1,2)$ tensor product properties, one may build, under some assumptions, an extension $\mathbf{S}$ of the $so(1,2)$ algebra going beyond the standard supersymmetric one. To do so, note first that the system J_0, J_+,J_- and Q_{s+n} obey the following commutation relations ($s=-1/k$):

$$\begin{aligned}[][J_0, Q_{s+n}] &= (s+n)Q_{s+n},\\ [J_+, Q_{s+n}] &= \sqrt{(2s+n)(n+1)}Q_{s+n+1},\\ [J_-, Q_{s+n}] &= \sqrt{(2s+n-1)n}Q_{s+n-1}.\end{aligned} \tag{3.9}$$

Similarly we have for the antiholomorphic sector:

$$\begin{aligned}[][\bar{J}_0, \bar{Q}_{s+n}] &= -(s+n)\bar{Q}_{s+n},\\ [\bar{J}_+, \bar{Q}_{s+n}] &= -\sqrt{(2s+n-1)n}\bar{Q}_{s+n-1},\\ [\bar{J}_-, \bar{Q}_{s+n}] &= -\sqrt{(2s+n)(n+1)}\bar{Q}_{s+n+1}.\end{aligned} \tag{3.10}$$

To close these commutations relations with the Q_s's through a kth order product one should fulfil some constraints.

(1) Tthe generalized algebra $\mathbf{S}$ we are looking for should be a generalization of what is known in two dimensions, i.e. a generalization of FSS.

(2) When the charge operator Q_{s+n} goes around other, say Q_{s+m}, it picks up a phase $\Phi = 2\mathrm{i}\pi/k$; i.e,

$$Q_{s+n}Q_{s+m} = \mathrm{e}^{\pm 2\mathrm{i}\pi s}Q_{s+m}Q_{s+n} + \ldots, \quad s = -\frac{1}{k}, \tag{3.11}$$

where the dots refer for possible extra charge operators of total J_0 eigenvalue $(2s+n+m)$. Equation (3.11) shows also that the algebra we are looking for has a $\mathbf{Z}_k$ graduation. Under this discrete symmetry, Q_{s+n} carries a $+1$ (mod$k$) charge while the $P_{0,\pm}$ energy momentum components have a zero charge mod$k$.

(3) The generalized algebra $\mathbf{S}$ should split into a bosonic B part and an anyonic A part and may be written as $\mathbf{S} = \oplus_{r=0}^{k-1} A_r = B \oplus_{r=1}^{k-1} A_r$. Since $A_n A_m \subset A_{(n+m)\ (\mathrm{mod}k)}$ one has

$$\begin{aligned}\{A_r \ldots A_r\}_k &\subset B,\\ [B,A] &\subset A,\\ [B,B] &\subset B.\end{aligned} \tag{3.12}$$

In these equations, $\{A_r \ldots A_r\}_k$ means the complete symmetrization of the k anyonic operators A_r and is defined as

$$\{A_{s_r} \ldots A_{s_r}\}_k = \frac{1}{k!}\sum_{\sigma\in\Sigma}(A_{s_{\sigma(1)}} \ldots A_{s_{\sigma(k)}}), \tag{3.13}$$

where the sum is taken over the k elements of the permutation group $\{1,\ldots,k\}$.

(4) The algebra $\mathbf{S}$ should obey generalized Jacobi identities. In particular we should have

$$\mathrm{ad}B\{A_{s_1} \ldots A_{s_k}\} = 0, \tag{3.14}$$

where B stands for the bosonic generators $J_{0,\pm}$ or $P_{0,\pm}$ of the Poincaré algebra. Using (3.12) to write $\{A_r \dots A_r\}_k$ as $\alpha_\mu P^\mu + \beta_\mu J^\mu$ where α and β are real constants; then putting this back into the above relation we find that $\{A_r \dots A_r\}_k$ is proportional to P_μ only. In other words, β_μ should be equal to zero; a property which is easily seen by taking $B = P_\mu$ in (3.14). Put differently the symmetric product of the $D_s^\pm$, denoted hereafter as $S^k[D_s^\pm]$, contains the space-time vector representation D_1 of $so(1,2)$ and so the primitive charge operators $Q_{-1/k}$ and $\bar{Q}_{1/k}$ obey

$$[J_0, (Q_{-1/k})^k] = -(Q_{-1/k})^k \sim P_-, \quad [J_-, (Q_{-1/k})^k] = 0. \tag{3.15}$$

Similarly we have

$$[\bar{J}_0, (\bar{Q}_{1/k})^k] = (\bar{Q}_{1/k})^k \sim P_+, \quad [\bar{J}_+, (\bar{Q}_{1/k})^k] = 0. \tag{3.16}$$

Moreover, acting on $(Q_{-1/k})^k$ by $\mathrm{ad}J_+^n$ and on $(\bar{Q}_{1/k})^k$ by $\mathrm{ad}\bar{J}_+^n$, one obtains

$$\begin{aligned} \mathrm{ad}J_+(Q_{-1/k})^k &\sim P_0, \\ \mathrm{ad}\bar{J}_-(\bar{Q}_{1/k})^k &\sim P_0, \\ \mathrm{ad}^2 J_+(Q_{-1/k})^k &\sim P_-, \\ \mathrm{ad}^2 \bar{J}_-(\bar{Q}_{1/k})^k &\sim P_+. \end{aligned} \tag{3.17}$$

In summary, starting from the $so(1,2)$ Lorentz algebra (3.1) and (3.2) and the two Verma modules HWR(I) and HWR(II) (3.3) and (3.4), one may build the following new extended symmetry:

$$\begin{aligned} P_\mp &= \{Q^\pm_{-\frac{1}{k}}, Q^\pm_{-\frac{1}{k}}, \dots, Q^\pm_{-\frac{1}{k}}\}_k, \\ \pm\mathrm{i}\sqrt{\frac{2}{k}} P_0 &= \left\{Q^\pm_{-\frac{1}{k}}, \dots, Q^\pm_{-\frac{1}{k}}, Q^\pm_{1-\frac{1}{k}}\right\}_k, \\ P_\pm &= -(k-1)\left\{Q^\pm_{-\frac{1}{k}}, \dots, Q^\pm_{-\frac{1}{k}}, Q^\pm_{1-\frac{1}{k}}, Q^\pm_{1-\frac{1}{k}}\right\}_k \\ &\quad \pm\mathrm{i}\sqrt{k-2}\left\{Q^\pm_{-\frac{1}{k}}, \dots, Q^\pm_{-\frac{1}{k}}, Q^\pm_{1-\frac{1}{k}}, Q^\pm_{2-\frac{1}{k}}\right\}_k, \\ 0 &= \left[J^\pm, \left[J^\pm, \left[J^\pm, \left(Q^\pm_{-\frac{1}{k}}\right)^k\right]\right]\right]. \end{aligned} \tag{3.18}$$

Equation (3.18) defines what we have been referring to as the RdTS algebra. For more details on this algebraic structure, see [1,23].

4 Links with BCFT on $\partial\mathrm{AdS}_3$

Here we would like to answer the question raised in the introduction concerning the link between RdTS supersymmetry and 2-dimensional conformal invariance. We have anticipated on the nature of this link by saying that RdST supersymmetry is expected to arise from appropriate deformations of 2-dimensional CFT's on the boundary of AdS_3. The appearance of the AdS_3 space in this analysis is due to the fact that this geometry has many relevant features for our present study. We give hereafter two useful properties regarding the space-time $SO(1,2)$ Lorentz group:

(a) In its euclidean representation, AdS_3 has an $SO(1,3)$ isometry group containing as a subgroup the $SO(1,2)$ Lorentz symmetry of the $(1+2)$ space-time we are interested in.
(b) The 2-dimensional AdS_3 boundary space may be realized as a 2-sphere on which may live boundary conformal field theories, which themselves have $so(1,2)$ projective subsymmetries that can be related to the above mentioned $so(1,2)$ Lorentz group.

Starting from these observations we want to show that the two so(1,2) modules HWR(I) and HWR(II), considered in the building of RdTS supersymmetry, are just special representations of the AdS_3 BCFT. To prove this relation in a comprehensive manner, let us first review briefly some elements of the AdS_3 geometry. The AdS_3 space is given by the hyperbolic coset manifold $Sl(2,C)/SU(2)$ which may be thought of as the 3-dimensional hypersurface ${H_3}^+$,

$$-{X_0}^2 + {X_1}^2 + {X_2}^2 + {X_3}^2 = -l^2, \tag{4.1}$$

embedded in flat $R^{1,3}$ with local coordinates X^0, X^1, X^2, X^3. This hypersurface describes a space with a constant negative curvature $(-1/l^2)$. The parameter l is chosen to be quantized in terms of the l_s fundamental string length units; i.e., $l = l_\mathrm{s} \times k$ where k is an integer to be interpreted later on as the Kac–Moody level of the $so_k(1,2)$ affine symmetry. To study the field theory on the boundary of AdS_3, it is convenient to introduce the following set of local coordinates of AdS_3:

$$\begin{aligned} \phi &= \log(X_0 + X_3)/l, \\ \gamma &= \frac{X_2 + \mathrm{i}X_0}{X_0 + \mathrm{i}X_3}, \\ \bar{\gamma} &= \frac{X_2 - \mathrm{i}X_1}{X_0 + \mathrm{i}X_3}. \end{aligned} \tag{4.2}$$

An equivalent description of the hypersurface is

$$\begin{aligned} \gamma &= \frac{r}{\sqrt{l^2 + r^2}} \mathrm{e}^{-\tau + \mathrm{i}\theta}, \\ \bar{\gamma} &= \frac{r}{\sqrt{l^2 + r^2}} \mathrm{e}^{-\tau - \mathrm{i}\theta}, \\ \phi &= \tau + 1/2 \log(1 + r^2/l^2), \\ r &= l\mathrm{e}^{\phi}\sqrt{\gamma\bar{\gamma}}, \\ \tau &= \phi - 1/2 \log(1 + \mathrm{e}^{2\phi}\gamma\bar{\gamma}), \\ \theta &= \frac{1}{2\mathrm{i}} \log(\gamma/\bar{\gamma}), \end{aligned} \tag{4.3}$$

where we have used the change of variables

$$\begin{aligned} X_0 &= X_0(r,\tau) = \sqrt{l^2 + r^2} \cosh\tau, \\ X_3 &= X_3(r,\tau) = \sqrt{l^2 + r^2} \sinh\tau, \\ X_1 &= X_1(r,\theta) = r\sin\theta, \\ X_2 &= X_2(r,\theta) = r\cos\theta. \end{aligned} \tag{4.4}$$

In the coordinates $(\phi, \gamma, \bar{\gamma})$, the metric of H_3^+ reads

$$\mathrm{d}s^2 = k(\mathrm{d}\Phi^2 + \mathrm{e}^{2\Phi}\mathrm{d}\gamma \mathrm{d}\bar{\gamma}). \tag{4.5}$$

Note that in the $(\phi, \gamma, \bar{\gamma})$ frame, the boundary of the euclidean AdS_3 corresponds to taking the field Φ to infinity. As shown in (4.3) and (4.4), this is a 2-sphere which is locally isomorphic to the complex plane parameterized by $(\gamma, \bar{\gamma})$.

Quantum field theory on the AdS_3 space is very special and has very remarkable features, governed by the Maldacena correspondence in the zero slope limit of string theory [24]. On this space it has been shown that bulk correlations functions of quantum fields find natural interpretations in the conformal field theory on the boundary of AdS_3 [9]. In algebraic language, this correspondence transforms world sheet symmetries of strings on AdS_3 into space-time infinite dimensional invariances on the boundary of AdS_3. In what follows we shall review some useful properties of strings on AdS_3 and $\partial\mathrm{AdS}_3$.

4.1 AdS_3–CFT correspondence

Strings propagating on the AdS_3 background are involved in the study of supersymmetric gauge theories with eight supercharges; in particular in the understanding of the Higgs and Coulomb branches near the moduli space singularity [25]. Strings on AdS_3 have rich symmetries; some of these turn out to be related to the problem we are studying. These symmetries, which may be classified into WS symmetries and space-time invariances, carry all relevent information one needs to know about the string dynamics on AdS_3. In what follows we want to give some useful relations regarding these two classes of symmetries. To work out explicit field theoretical realizations of these symmetries, we start by recalling that in the presence of the Neveu–Schwarz $B_{\mu\nu}$ field with euclidean world sheet parameterized by $(z, \bar{z})$, the dynamics of the bosonic string on AdS_3 is described by the following classical lagrangian:

$$L = k[\partial\Phi\bar{\partial}\Phi + \mathrm{e}^{2\Phi}\partial\gamma\partial\bar{\gamma}]. \tag{4.6}$$

In this equation ∂ and $\bar{\partial}$ stand for derivatives with respect to z and $\bar{z}$, respectively. Introducing the two auxiliary variables β and $\bar{\beta}$, the above equation may be put into the following convenient form:

$$L^{'} = k^2(\partial\Phi\bar{\partial}\Phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} - \mathrm{e}^{-2\Phi}\beta\bar{\beta}). \tag{4.7}$$

The equations of motion of the various fields one gets from (4.7) read

$$\begin{aligned} \partial\bar{\partial}\Phi - 2\beta\bar{\beta}\mathrm{e}^{-2\Phi} &= 0, \\ \bar{\partial}\gamma - \beta\mathrm{e}^{-2\Phi} &= 0, \\ \partial\bar{\gamma} - \bar{\beta}\mathrm{e}^{-2\Phi} &= 0, \\ \partial\bar{\beta} = \bar{\partial}\beta &= 0. \end{aligned} \tag{4.8}$$

String dynamics on the boundary of AdS_3 is obtained from the previous bulk equations by taking the limit where Φ goes to infinity. This gives

$$\begin{aligned} \partial\bar{\partial}\Phi &= 0, \\ \bar{\partial}\gamma = \partial\bar{\gamma} &= 0, \\ \partial\bar{\beta} = \bar{\partial}\beta &= 0. \end{aligned} \tag{4.9}$$

The WS fields Φ, γ and $\bar{\gamma}$, which had general expressions in the bulk, become now holomorphic on the boundary of AdS_3 and describe a BCFT. Note that consistency of quantum mechanics of the string propagating in space-time requires that the target space should be $\mathrm{AdS}_3 \times N$, where N is a $(3 + n)$-dimensional compact manifold. To fix the ideas, N may be thought of as $S^3 \times T^n$ with $n = 20$ for the bosonic string and $n = 4$ for superstrings. We shall consider hereafter both string and superstring cases. Given the large number of relations one may write down, we shall use however a strategy in which we give the strictly necessary results. Thus our plan in what follows is: First, we describe some algebraic features of the WS invariance; then we make a pause to give a complement on FSS using the spectral flow of $N = 2$ and $N = 4$ conformal invariance, after which we return to complete space-time symmetries on the boundary of AdS_3, and finally we give our results.

4.2 WS symmetries

World sheet invariances include affine Kac–Moody, Virasoro symmetries and their extensions. For a bosonic string propagating on $\mathrm{AdS}_3 \times S^3 \times T^{20}$, we have the following:

A Three kinds of WS affine Kac–Moody invariances

(a) A level $(k-2)$ $sl(2) \times \bar{sl}(2)$ invariance coming from the string propagation on AdS_3. This invariance is generated by the conserved currents $J^q_{sl(2)}$ and $\bar{J}^q_{sl(2)}; q = 0, \pm 1$. In terms of the WS fields $\Phi, \gamma, \bar{\gamma}, \beta$ and $\bar{\beta}$ of (4.7), the field theoretical realization of these currents is given by the Wakimoto representation:

$$\begin{aligned} J^-(z) &= \beta(z), \\ J^+(z) &= \beta\gamma^2 + \sqrt{2(k-2)}\gamma\partial\Phi + k\partial\gamma, \\ J^0(z) &= \beta\gamma + 1/2\sqrt{2(k-2)}\partial\Phi, \\ \bar{J}^-(\bar{z}) &= \bar{\beta}, \\ \bar{J}^0(\bar{z}) &= \bar{\beta}\bar{\gamma} + 1/2\sqrt{2(k-2)}\partial\Phi, \\ \bar{J}^+(\bar{z}) &= \bar{\beta}\bar{\gamma}^2 + \sqrt{2(k-2)}\bar{\gamma}\partial\Phi + k\partial\bar{\gamma}. \end{aligned} \tag{4.10}$$

(b) A level $(k+2)$ invariance coming from the string propagation on S^3. The conserved currents are $J^q_{su(2)}$ and $\bar{J}^q_{su(2)}$. The WS field theoretical realization of these currents is given by the level $(k + 2)$ WZW $su(2)$ model [26].

(c) A $u(1)^{20} \times \bar{u}(1)^{20}$ invariance coming from the torus T^{20}. This symmetry is generated by 20 $U(1)$ Kac–Moody currents $J^i_{u(1)}; i = 1, \ldots, 20$.

B WS Virasoro symmetry

This symmetry, which splits into holomorphic and antiholomorphic sectors, is given by the Suggawara construction using quadratic Casimirs of the previous WS affine Kac–Moody algebras. For the holomorphic sector, the WS Virasoro currents of a bosonic string on $\mathrm{AdS}_3 \times S^3 \times T^{20}$ are

(a) String on AdS_3:

$$T^{\mathrm{WS}}_{sl(2)} = \frac{1}{(k-2)}[(J^0_{sl(2)})^2 - (J^1_{sl(2)})^2 - (J^2_{sl(2)})^2]. \quad (4.11)$$

(b) String on S^3:

$$T^{\mathrm{WS}}_{su(2)} = \frac{1}{(k+2)}[(J^0_{su(2)})^2 + (J^1_{su(2)})^2 + (J^2_{su(2)})^2]. \quad (4.12)$$

(c) String on T^{20}:

$$T^{\mathrm{WS}}_{u(1)} = \sum_{i=1}^{20}[J^i_{u(1)}]^2. \quad (4.13)$$

Similar quantities are also valid for the antiholomorphic sector of the conformal invariance. Note that the total WS energy momentum tensor $T^{\mathrm{WS}}_{\mathrm{tot}}$ is given by the sum of $T^{\mathrm{WS}}_{sl(2)}, T^{\mathrm{WS}}_{su(2)}$ and $T^{\mathrm{WS}}_{u(1)}$, (4.11), (4.12) and (4.13).

In the case of a superstring propagating on $\mathrm{AdS}_3 \times S^3 \times T^4$, the above conserved currents are slightly modified by the adjunction of extra terms due to contributions of WS fermions. If we denote by $\Psi^A_{sl(2)}, \Psi^a_{su(2)}$ and $\Psi^i_{u(1)}$ the AdS_3, S^3 and T^4 fermions, the WS theory has a $N = 1$ superconformal theory generated by

$$\begin{aligned} T(z) = &\frac{1}{k}[(J^A_{sl(2)}J_{sl(2),A} - \Psi^A_{sl(2)}\partial\Psi_{sl(2),A}) \\ &+(J^a_{su(2)}J_{su(2),a} - \Psi^a_{su(2)}\partial\Psi_{su(2),a})] \\ &+1/2\sum_{i=1}^{4}(J^i_{u(1)}J^i_{u(1)} - \Psi^i_{u(1)}\partial\Psi^i_{u(1)}), \\ G(z) = &\frac{2}{k}\left[\Psi^A_{sl(2)}J_{sl(2),A} - \frac{\mathrm{i}}{3k}\epsilon_{ABC}\Psi^A_{sl(2)}\Psi^B_{sl(2)}\Psi^C_{sl(2)}\right] \\ &+\frac{2}{k}\left[\Psi^a_{su(2)}J_{su(2),a} - \frac{\mathrm{i}}{3k}\epsilon_{abc}\Psi^a_{su(2)}\Psi^b_{su(2)}\Psi^c_{su(2)}\right] \\ &+\sum_{i=1}^{4}\Psi^i_{u(1)}\partial J^i_{u(1)}. \end{aligned} \quad (4.14)$$

Note that to get a space-time supersymetric vacuum, one should enhance the previous $N = 1$ superconformal WS invariance to a $N = 2$ conformal symmetry [27]. This requires the existence of a conserved U(1) current in the world sheet theory under which G splits in two parts G^+ and G^- with charges $+1$ and -1, respectively. Skipping the details and denoting by $G^\pm_r$ the modes of the $G^\pm(z)$ $N = 2$ fermions currents, the $N = 2$ $U(1)$ superconformal algebras read

$$\begin{aligned} [G^-_r, G^+_s] &= 2L_{r+s} - (r-s)J_{r+s} \\ &\quad + (c/3)(r^2 - 1/4)\delta_{r+s,0}, \\ [L_n, L_m] &= (n-m)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\ [L_n, G^\pm_r] &= \left(\frac{n}{2} - r\right)G^\pm_{n+r}, \\ [L_n, J_m] &= -mJ_{m+n}, \\ [J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0}, \\ [J_n, G^\pm_r] &= \pm G^\pm_{n+r}, \end{aligned} \quad (4.15)$$

where the r and s modes take half odd integer values for the Neveu–Schwarz (NS) sector and integer ones for the Ramond (R) sector. Before going ahead we would like to make a pause in order to give some relevant features of these algebras. This pause is motivated by the two following items. First the $N = 2$ NS and R conformal algebras have a spectral flow which we want to use in order to complete the study of Sect. 2 on FSS by giving a new result. Second space-time symmetry of the superstring on $\mathrm{AdS}_3 \times S^3 \times T^4$ has a $N = 4$ superconformal invariance which has a spectral flow of the same nature as for $N = 2$ U(1) conformal invariance. Like for the FSS case, the spectral flow of the $N = 2$ and $N = 4$ conformal invariances may also be used to study RdTS supersymmetry.

5 FSS and spectral flow

In Sect. 2, we have defined FSS as a hidden finite dimensional invariance which survives after integrable deformations of critical models such as the thermal deformation of Z_N models; see (2.11) and (2.12). There, we exposed a method for deriving FSS algebras from parafermionic invariance. In the present section we want to complete the study of Sect. 2 by giving a new way for obtaining FSS using topological field theory ideas [28]. This method is based on using an appropriate choice of the parameter η of the spectral flow of $N = 2$ and $N = 4$ superconformal theories. We will also take the opportunity in analysing the spectral flow of $N = 2$ and $N = 4$ conformal symmetries to make a comment on the recent proposal of [29], where a new construction of fractional supersymmetric algebras was derived by using infinite dimensional modules of Lie algebras.

For a start, recall that due to boundary conditions of fermions, the 2-dimensional $N = 2$ ($N = 4$) superconformal algebra has two sectors: the Neveu–Schwarz (NS) sector and Ramond (R) sector. These two sectors are not completely independent since they may be related by a continuous spectral flow as shown here:

$$\begin{aligned} U_\theta L_n U_\theta^{-1} &= L_n + \theta J_n + c/6\theta^2\delta_{n,0}, \\ U_\theta J_n U_\theta^{-1} &= J_n + c/3\theta\delta_{n,0}, \\ U_\theta G^+_r U_\theta^{-1} &= G^+_{r+\theta}, \\ U_\theta G^-_r U_\theta^{-1} &= G^-_{r-\theta}, \end{aligned} \quad (5.1)$$

for $N = 2$ theories, and

$$\begin{aligned} T^3_n(\eta) &= T^3_n(0) - \frac{\eta k p}{2}\delta_{n,0}, \\ T^{\pm}_{n\pm\eta}(\eta) &= T^{\pm}_n(0), \\ Q^1_{n+n/2}(\eta) &= Q^1_n(0), \\ Q^2_{n-\eta/2}(\eta) &= Q^2_n(0), \\ L_n(\eta) &= L_n(\eta) - \eta T^3_n(0) + \eta^2\left(\frac{kp}{4}\right)\delta_{n,0}, \end{aligned} \tag{5.2}$$

for $N = 4$ superconformal ones. The variable η is the parameter of the spectral flow. Equations (5.1) and (5.2) mean that 2-dimensional $N = 2$ ($N = 4$) superconformal algebras then have a continuous one parameter sector interpolating between the NS and R algebras. This interpolating sector is generated by mode operators $G^{\pm}_{r\pm\eta}$ and $\bar{G}^{\pm}_{r\pm\eta}$ carrying shifted values of the L_0 and the $U(1)$ charge operators. For a generic value of η, the commutation relations of the $N = 2$ superconformal algebra in two dimensions read

$$\begin{aligned} \{G^+_{r+\eta}\bar{G}^-_{s-\eta}\} &= 2L_{r+s} - (r - s + 2\eta)J_{r+s} \\ &\quad +(c/3)((r+\eta)^2 - 1/4)\delta_{r+s,0}, \\ [L_n, L_m] &= (n-m)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}, \\ [L_n, G^{\pm}_{r\pm\eta}] &= \left(\frac{n}{2} - r \mp \eta\right)G^{\pm}_{n+r\pm\eta}, \\ [L_n, J_m] &= -mJ_{m+n}, \\ [J_m, J_n] &= \frac{c}{3}m\delta_{m+n,0}, \\ [J_n, G^{\pm}_{r+\eta}] &= \pm G^{\pm}_{n+r+\eta}, \\ \{G^+_{r+\eta}\bar{G}^+_{s-\eta}\} &= 0, \\ \{G^-_{r+\eta}\bar{G}^-_{s-\eta}\} &= 0. \end{aligned} \tag{5.3}$$

Similar equations may be written down for the $N = 4$ case. Equations (5.3) define a continuous one family parameter superconformal algebra to which we shall refer below as the η sector and we denote it $[(1-2\eta)\mathrm{NS}, 2\eta\mathrm{R}]$. For $\eta = 0$, one discovers the NS algebra and for $\eta = 1/2$ one gets the R algebra. For η ranging between zero and $1/2$, one has the twisted sector. The $[(1-2\eta)\mathrm{NS}, 2\eta\mathrm{R}]$ twisted conformal algebra plays a crucial role in topological field theories [28, 30] and allows one to make spectacular transformations such as modifying the spins of the WS field operators by making appropriate choices of η. Taking the spectral parameter $\eta = 1/2$, a fermion transforms into a boson (scalar or vector) while taking $\eta = 1/k$, $k > 2$; it becomes a WS parafermion of spin $(1 \pm \eta)$ depending on the $U(1)$ charge of the initial fermion. Putting back $\eta = 1/k$ into (5.3), one gets amongst others

$$2P_{-1} = \{G^+_{-(k-1)/k}, G^-_{-1/k}\} + \{G^+_{-1/k}, G^-_{-(k-1)/k}\}, \tag{5.4}$$

together with

$$\begin{aligned} 0 &= \{G^{\pm}_{-1/k}, G^{\pm}_{-1/k}\}, \\ 0 &= \{G^{\pm}_{-(k-1)/k}, G^{\pm}_{-(k-1)/k}\}. \end{aligned} \tag{5.5}$$

Now comparing these relations with (2.15), which we obtained by a thermal deformation of the Z_k parafermionic invariance, one discovers that they are quite similar. Equation (5.4) gives just a linearization form of FSS which coincides with (2.15) by setting $k = 3$. Moreover (5.5) show that $G^{\pm}_{-1/k}$ are anticommuting operators in agreement with the result of [31]. Furthermore starting from (5.4) and (5.5) and following the reasoning of Sect. 2 which led to the derivation of (2.15), one sees that it is possible to reinterpret the minus charge carried by $G^-_{(1-k)/k}$ as a Z_k charge. So $G^-_{(1-k)/k}$ may be viewed as a composite operator given by the product of $(k-1)$ and $G^+_{-1/k}$. This property is also supported by the fact that the $N = 2$ superconformal currents have mode expansion operators with twisted values. We have

$$G^{\pm}(z_1)\Phi_m(z_2) = \sum z_{12}^{n-1\mp p/k} G^{\pm}_{-n\pm(p\pm1)/k}\Phi_m(z_2). \tag{5.6}$$

Using these modes operators, one may write for $k = 3$ the following relations

$$G^-_{-\frac{2}{3}} = G^+_{-\frac{2}{3}}G^+_0. \tag{5.7}$$

The spectral flow of $N = 2$ superconformal theories gives then another way to build FSS algebras. In this regard, it is interesting to note that this spectral flow analysis might also be used to rederive the so-called FSUSY algebras considered recently in [29]. We suspect that the fractional quantum numbers considered in [29] when deriving FSUSY from special Verma modules of finite dimensional Lie algebras g could be rederived by taking fractional values of the spectral parameters η of the corresponding Kac–Moody algebra $\hat{g}$. Recall in passing that under the spectral flow, the step generators J^{α}_n and the Cartan ones H^i_n of $\hat{g}$ transform as

$$\begin{aligned} J^{\alpha}_n &\to J^{\alpha}_{n+\eta v.\alpha}, \\ H^i_n &\to H^i_n + k\eta v^i\delta_{n,0}, \end{aligned} \tag{5.8}$$

where α are the roots of $\hat{g}$ and v is a weight vector. This transformation shifts the eigenvalues of the H^i_n's Cartan charge operators of $\hat{g}$. By an appropriate choice of the free parameters in the shifted weight $(2k\eta/\alpha^2)\alpha^i v^i$ of $(2/\alpha^2)\alpha^i H^i_0$, one recovers the fractionality property of the quantum numbers used in the construction of FSUSY algebras [29]. This issue will be exhibited in more detail when we have a future occasion [32]. Now we turn to our main topic.

6 Space-time invariance

To analyze the space-time infinite dimensional symmetries on the boundary of AdS_3, one may follow the same strategy that we have used for the study of WS invariances. First identify the space-time affine Kac–Moody symmetries and then consider the space-time conformal invariance and eventually the Casimirs of higher ranks. In this section we shall simplify a little bit the analysis of space-time invariance and focus our attention on the conformal symmetry on $\partial(\mathrm{AdS}_3)$. Some specific features of space-time Kac–Moody symmetries will also be given in due time.

We begin by noting that space-time infinite invariances on the boundary of AdS_3 are intimately linked to the WS ones. For the case of a superstring propagating on $\mathrm{AdS}_3 \times S^3 \times T^4$, we have already shown that there are various kinds of WS symmetries coming from the propagation on AdS_3, S^3 and T^4 respectively. In the ϕ infinite limit, we want to show that one may use these WS symmetries to build new space-time ones.

6.1 Conformal invariances

First of all, note that the global part of the WS $SO(1,2) \times \bar{SO}(1,2)$ affine invariance of a bosonic string on AdS_3, generated by J_0^q and $\bar{J}_0^q; q = 0, \pm 1$, may be realized in different ways. A tricky way, which turns out to be crucial in building space-time conformal invariance, is given by the Wakimoto realization [22]. Classically, this representation reads in terms of the local coordinates $(\Phi, \gamma, \bar{\gamma})$

$$
\begin{aligned}
J_0^0 &= \gamma \partial/\partial\gamma - 1/2 \partial/\partial\gamma, \\
J_0^- &= \partial/\partial\gamma, \\
J_0^+ &= \gamma^2 \partial/\partial\gamma - \gamma \partial/\partial\Phi - \mathrm{e}^{-2\Phi} \partial/\partial\gamma.
\end{aligned}
\tag{6.1}
$$

Similar relations are also valid for $\bar{J}_0^q$; they are obtained by substituting γ by $\bar{\gamma}$. Quantum mechanically, the charge operators J_0^q and $\bar{J}_0^q$ are given in terms of the Laurent mode operators of the quantum fields $\Phi, \gamma, \bar{\gamma}, \beta$ and $\bar{\beta}$ by using (4.10) and performing the Cauchy integrations

$$
\begin{aligned}
J_0^q &= \int \frac{\mathrm{d}z}{2\mathrm{i}\pi} J^q(z), \\
\bar{J}_0^q &= \int \frac{\mathrm{d}\bar{z}}{2\mathrm{i}\pi} \bar{J}^q(z).
\end{aligned}
\tag{6.2}
$$

To build the space-time conformal invariance on the AdS_3 boundary, we proceed by the following steps. First suppose that there exists really a conformal symmetry on the boundary of AdS_3 and denote the space-time Virasoro generators by $\mathcal{L}_n$ and $\bar{\mathcal{L}}_n, n \in Z$. The $\mathcal{L}_n$ and $\bar{\mathcal{L}}_n$, which should not be confused with the WS conformal mode generators, obviously satisfy the left and right Virasoro algebras. We can write

$$
\begin{aligned}
[\mathcal{L}_n, \mathcal{L}_m] &= (n-m)\mathcal{L}_{n+m} + c/12 n(n^2-1)\delta_{n+m}, \\
[\bar{\mathcal{L}}_n, \bar{\mathcal{L}}_m] &= (n-m)\bar{\mathcal{L}}_{n+m} + \bar{c}/12 n(n^2-1)\delta_{n+m}, \\
[\mathcal{L}_n, \bar{\mathcal{L}}_m] &= 0.
\end{aligned}
\tag{6.3}
$$

The second step is to solve these equations by using the string WS fields $(\Phi, \gamma, \bar{\gamma})$ on AdS_3. To do so, it is convenient to divide the above equations into two blocks. The first block corresponds to setting $n = 0, \pm 1$ in the generators $\mathcal{L}_n$ and $\bar{\mathcal{L}}_n$ of (6.3). It describes the anomaly free projective subsymmetry the Virasoro algebra. The second block concerns the generators associated with the remaining values of n.

On the boundary of AdS_3 obtained by taking the infinite limit of the Φ field, one solves the projective subsymmetry by the natural identification of $\mathcal{L}_q$ and $\bar{\mathcal{L}}_q$; $q = 0, \pm 1$ with the zero modes of the WS $so(1,2) \times \bar{so}(1,2)$ affine invariance. In other words we have

$$
\begin{aligned}
\mathcal{L}_q &= -\int \frac{\mathrm{d}z}{2\mathrm{i}\pi} J^q(z) = -J_0^q; \quad q = 0, \pm 1, \\
\bar{\mathcal{L}}_q &= \int \frac{\mathrm{d}\bar{z}}{2\mathrm{i}\pi} \bar{J}^q(z) = -\bar{J}_0^q; \quad q = 0, \pm 1.
\end{aligned}
\tag{6.4}
$$

Note that on the AdS_3 boundary, viewed as a complex plane parameterized by $(\gamma, \bar{\gamma})$, the charge operators J_0^- $(\mathcal{L}_{-1})$ and $\bar{J}_0^-(\bar{\mathcal{L}}_{-1})$ taken in the Wakimoto representation coincide respectively with the translation operators P_- and $\bar{P}_+$:

$$
\begin{aligned}
P_- &= \mathcal{L}_- = \partial/\partial\gamma, \\
P_+ &= \bar{\mathcal{L}}_- = \partial/\partial\bar{\gamma}.
\end{aligned}
\tag{6.5}
$$

Equations (6.4) and (6.5) are interesting; they establish a link between the $\mathcal{L}_-$ and $\bar{\mathcal{L}}_-$ constants of motion of the boundary conformal field theory on AdS_3 on the one hand and the $P_-(=P)$ and the $P_+(=\bar{P})$ translation generators of the ST extension of the $so(1,2)$ algebra on the other hand. We will turn to these relations in the discussion of Sect. 7.

To get the rigorous solution of the remaining Virasoro charge operators $\mathcal{L}_n$ and $\bar{\mathcal{L}}_n$, one has to work hard. This is a lengthy and technical calculation which has been done in [10] in connection with the study of the $\mathrm{D}_1/\mathrm{D}_5$-brane system. Later on we shall give some indications on this method; for the time being we shall use an economic path to work out the solution. This is a less rigorous but tricky way to get the same result. This method is based on trying to extend the $\mathcal{L}_n$ and $\bar{\mathcal{L}}_n; n = 0, \pm 1$ projective solution to arbitrary integers n using properties of the string WS fields near the boundary, dimensional arguments and similarities with the photon vertex operator in three dimensions. Indeed using the holomorphic property of γ and $\bar{\gamma}$ (4.9) as well as space-time dimensional arguments,

$$
\begin{aligned}
&[\gamma] = -1; \quad J^0_{sl(2)} = 0, \\
&J^-_{sl(2)} = 1; \quad J^+_{sl(2)} = -1,
\end{aligned}
\tag{6.6}
$$

it is not difficult to check that the following $\mathcal{L}_n(\bar{\mathcal{L}}_n)$ expressions are good candidates:

$$
\begin{aligned}
\mathcal{L}_n = \int \frac{\mathrm{d}z}{2\mathrm{i}\pi} \Big[& a_0 \gamma^n J^0_{sl(2)} - \frac{a_-}{2} \gamma^{n+1} J^-_{sl(2)} \\
& + \frac{a_+}{2} \gamma^{n-1} J^+_{sl(2)} \Big],
\end{aligned}
\tag{6.7}
$$

and a similar relation for $\bar{\mathcal{L}}_n$. To get the a_i coefficients, one needs to impose constraints which may be obtained by the using results of a BRST analysis in QED in three dimensions. Following [9], the right constraints one has to impose on the a_i's are

$$na_0 + (n+1)a_- + (n-1)a_+ = 0,$$
$$J^0\gamma - (1/2)J^-\gamma^2 - (1/2)J^+ = 0. \quad (6.8)$$

The solution of the first constraint of these equations reproducing the projective generators (6.4) is as follows:

$$a_0 = (n^2-1),$$
$$a_- = n(n-1),$$
$$a_+ = n(n+1). \quad (6.9)$$

Moreover using the second constraint of (6.8) to express $J^+_{sl(2)}(z)$ in terms of $J^0_{sl(2)}(z)$ and $J^-_{sl(2)}(z)$, then putting this back into (6.7), we find

$$L_n = \int \frac{\mathrm{d}z}{2\mathrm{i}\pi}[-(n+1)\gamma^n J^0_{sl(2)} + n\gamma^{n+1}J^-_{sl(2)}]. \quad (6.10)$$

Equations (6.4) and (6.10) define the space-time Virasoro algebra on the boundary of AdS_3.

6.2 Comments

Having build the L_n space-time Virasoro generators, one may be interested in determining the space-time energy momentum tensors $T(\gamma)$ and $\bar{T}(\bar{\gamma})$ of the BCFT on AdS_3. It turns out that the right form of the space-time energy momentum tensor depends moreover on auxiliary complex variables $(y,\bar{y})$ so that the space-time energy momentum tensor has now two arguments; i.e. $T = T(y,\gamma)$ and $\bar{T} = \bar{T}(\bar{y},\bar{\gamma})$. Following [10], $T(y,\gamma)$ and $\bar{T}(\bar{y},\bar{\gamma})$ read

$$T(y,\gamma) = \int \frac{\mathrm{d}z}{2\mathrm{i}\pi}\left[\frac{\partial_y J(y,\gamma)}{(y-\gamma)^2} - \frac{\partial^2{}_y J(y,\gamma)}{(y-\gamma)}\right],$$
$$\bar{T}(\bar{y},\bar{\gamma}) = \int \frac{\mathrm{d}\bar{z}}{2\mathrm{i}\pi}\left[\frac{\partial_{\bar{y}} J(\bar{y},\bar{\gamma})}{(\bar{y}-\bar{\gamma})^2} - \frac{\partial^2{}_{\bar{y}} J(\bar{y},\bar{\gamma})}{(\bar{y}-\bar{\gamma})}\right], \quad (6.11)$$

where the currents $J(y,\gamma)$ and $J(\bar{y},\bar{\gamma})$ are given by

$$J(y,\gamma) = -J^+(y,\gamma) = 2yJ^0(\gamma) - J^+(\gamma) - y^2J^-(\gamma). \quad (6.12)$$

In connection with these equations, it is interesting to note that the conserved currents $J^q(y,\gamma)$ and $J^q(\bar{y},\bar{\gamma})$ are related to the WS affine Kac–Moody ones on AdS_3 as follows:

$$J^+(y,\gamma) = \mathrm{e}^{-yJ_0^-}J^+(\gamma)\mathrm{e}^{yJ_0^-}$$
$$= J^+(\gamma) - 2yJ^0(\gamma) + y^2J^-(\gamma),$$
$$J^0(y,\gamma) = \mathrm{e}^{-yJ_0^-}J^0(\gamma)\mathrm{e}^{yJ_0^-}$$
$$= J^0(\gamma) - yJ^-(\gamma) = -\frac{1}{2}\partial_z J^+(y,\gamma),$$
$$J^-(y,\gamma) = \mathrm{e}^{-yJ_0^-}J^-(\gamma)\mathrm{e}^{yJ_0^-}$$
$$= J^-(\gamma) = \frac{1}{2}\partial_z^2 J^+(y,\gamma). \quad (6.13)$$

and analogous equations for $J^q(\bar{y},\bar{\gamma})$. Putting (6.12) back into (6.11) and expanding in a power series of γ/y, one discovers the L_n space-time Virasoro generators given by (6.10). The second comment we want to make concerns the building of space-time affine Kac–Moody symmetries out of the WS ones. Staring from WS conserved currents $E^a_{\mathrm{WS}}(z)$, which may be thought of as $J^q_{sl(2)}(z)$, and going to the boundary of AdS_3, the corresponding space-time affine Kac–Moody currents $E^a_{\text{space-time}}(y,\gamma)$ read

$$E^a_{\text{space-time}}(y,\gamma) = \oint \frac{\mathrm{d}z}{2\mathrm{i}\pi}\left[\frac{E^a_{ws}(z)}{(y-\gamma(z))}\right]. \quad (6.14)$$

Expanding this equation in powers of y/γ or γ/y, one gets the space-time affine Kac–Moody modes:

$$E_n^{a,\text{space-time}} = \oint \frac{\mathrm{d}z}{2\mathrm{i}\pi}[E^a_{\mathrm{WS}}(z)\gamma^n]. \quad (6.15)$$

The third comment we want to make concerns superstrings on $AdS_3 \times S^3 \times T^4$. In addition to the bosonic sector, there are moreover contributions coming from the WS fermions $\psi_{\mathrm{WS}}(z)$. On the AdS_3 space for which the WS fermions $\psi^q_{\mathrm{WS}}(z)$, $q = 0, \pm$, transform in the $SO(1,2)$ adjoint, the total level k $SO(1,2)$ currents $J^q_{sl(2),\mathrm{Total}}(z)$ now have two contributions: a level $(k+2)$ bosonic current $J^q_{sl(2),\mathrm{Bose}}(z)$ and a level (-2) fermionic current $J^q_{sl(2),\mathrm{Fermi}}$ (z). The same construction may also be made for both S^3 and T^4. Note finally that in the limit that ϕ goes to infinity, the space-time conformal symmetry of a superstring propagating on $AdS_3 \times S^3 \times T^4$ forms a $N = 4$ conformal invariance.

7 Discussion and conclusion

So far, we have learned that on $AdS_3 \times N^d$ may live various boundary conformal field theories depending on the choice of the d-dimensional compact manifold N^d. In the case of critical models of (super-) strings propagating on $AdS_3 \times N^d$, we have studied two examples:

(i) N^d is given by the T^{23} torus;
(ii) N^d is given by $S^3 \times T^4$.

The first example describes a bosonic BCFT while the second one describes a $N = 4$ BCFT. One may also consider other choices of N^d and build other BCFT's.

If one forgets about the string dynamics as well as the nature of the compact manifold N and just retains that on $\partial(AdS_3)$ lives a conformal structure, one may consider its highest weight representations, which read in general

$$[L_0, \Psi_{h,\bar{h}}] = h\Psi_{h,\bar{h}},$$
$$[L_n, \Psi_{h,\bar{h}}] = 0; \quad n \geq 1,$$
$$[\bar{L}_0, \Psi_{h,\bar{h}}] = \bar{h}\Psi_{h,\bar{h}},$$
$$[\bar{L}_n, \Psi_{h,\bar{h}}] = 0; \quad n \geq 1,$$
$$[cI, \Psi_{h,\bar{h}}] = c\Psi_{h,\bar{h}}. \quad (7.1)$$

In these relations, the $\Psi_{h,\bar{h}}$'s are conformal field operators living on $\partial(AdS_3)$; their corresponding Virasoro primary states $|h,\bar{h}\rangle$ are given by $|h,\bar{h}\rangle = \Psi_{h,\bar{h}}(0,0)|0\rangle$. A priori the central charge c and the conformal weights h and $\bar{h}$ of these representations are arbitrary. However, requiring unitary conditions, the parameters c, h and $\bar{h}$ are subject to constraints which become stronger if one imposes extra symmetries, such as supersymmetry or parafermionic invariance. Under appropriate assumptions, one may also end with a finite closed set $\{\Psi_{h_i,\bar{h_i}}\}$, $i=1,2,...$, of conformal field operators; i.e.

$$[\Psi_{h_i,\bar{h_i}}][\Psi_{h_j,\bar{h_j}}] = C_{ij}^k[\Psi_{h_k,\bar{h_k}}], \tag{7.2}$$

where $[\Psi_{h_i,\bar{h_i}}]$ stands for conformal blocks, $[F][G]$ for the OPE, the operator product expansion, and where the C_{ij}^k's are the structure constants of the fusion algebra.

Having these details in mind, one may also build the field descendants $\Psi_{(h+n,\bar{h}+\bar{n})}$ from the $\Psi_{h,\bar{h}}$ primary ones as follows:

$$\Psi_{(h+n,\bar{h}+\bar{n})} = \sum_{\substack{n=\sum\alpha_i n_i\\ \bar{n}=\sum\beta_j n_j}} \lambda_{\{\alpha_i\}\{\beta_j\}}(\Pi_i L_{-n_i}^{\alpha i})(\Pi_j \bar{L}_{-n_j}^{\beta j})\Psi_{h,\bar{h}}, \tag{7.3}$$

where the α_i's and β_j's are positive integers and $\lambda_{\alpha\beta}$ are C-numbers which we use to denote the collective coefficients $\lambda_{\{\alpha_i\}\{\beta_j\}}$ of the decomposition (7.2). They satisfy the following relations:

$$\begin{aligned}
[L_0,\Psi_{(h+n,\bar{h}+\bar{n})}] &= (h+n)\Psi_{(h+n,\bar{h}+\bar{n})},\\
[L_\pm,\Psi_{(h+n,\bar{h}+\bar{n})}] &= a_\pm(h,n)\Psi_{h\pm n,\bar{h}\pm\bar{n}},\\
[\bar{L}_0,\Psi_{(h+n,\bar{h}+\bar{n})}] &= (\bar{h}+\bar{n})\Psi_{h+n,\bar{h}+\bar{n}},\\
[\bar{L}_\pm,\Psi_{(h+n,\bar{h}+\bar{n})}] &= \bar{a}_\pm(\bar{h},\bar{n})\Psi_{h\pm n,\bar{h}\pm\bar{n}},
\end{aligned} \tag{7.4}$$

where $a_\pm(h,n)$ and $\bar{a}_\pm(\bar{h},\bar{n})$ are normalization factors. Making an appropriate choice of the $\lambda_{\alpha\beta}$ coefficients and taking the $a_\pm(h,n)$ and $\bar{a}_\pm(h,\bar{n})$ coefficients as given by

$$\begin{aligned}
a_-(h,n) &= \sqrt{(2h+n)(n+1)},\\
a_+(h,n) &= \sqrt{(2h+n-1)n},
\end{aligned} \tag{7.5}$$

one can get the two $so(1,2)$ modules used in building RdTS supersymmetry. Note that the descendant fields $\Psi_{h+n,\bar{h}+\bar{n}}$ are also eigenfunctions of the spin $(L_0-\bar{L}_0)$ and conformal scale $(L_0+\bar{L}_0)$ operators of eigenvalues $s=[(h-\bar{h})+(n-\bar{n})]$ and $\Delta=[(h+\bar{h})+(n+\bar{n})]$ respectively. Following the analysis of Sect. 2 and using conformal fields fusion rules as well as the mode expansion of the $\Psi_{h,\bar{h}}$ conformal field operator, in particular developments similar to (2.3), (2.4) and (2.8), one can build conserved charge operators carrying fractional values and work out the corresponding generalized supersymmetric algebra.

On the boundary of AdS_3, they may equally well live other structures such as affine symmetries and superconformal invariances going beyond the usual bosonic ones. Some of these structures were discussed in some detail throughout Sects. 4, 5 and 6; in particular those structures with direct relevance to the present study, namely the conformal structures having $N=2$ and $N=4$ supersymmetry. Using the spectral flows (5.1) and (5.2) of the $N=2$ and $N=4$ conformal algebras given by (4.15) and (5.3), the Wakimoto realization of the $Sl(2,R)$ affine Kac–Moody symmetry and topological field theoretical ideas, we have shown by an explicit analysis that here also RdTS supersymmetry may be interpreted as a specific deformation of the boundary conformal invariance on AdS_3, showing once more that RdTS invariance has much to do with the conformal structure on ∂AdS_3.

We conclude this study by saying that the RdTS extension of Poincaré invariance in $(1+2)$ dimensions we studied in this paper is a special kind of FSS algebra. Like for FSS invariances, the RdTS generalized algebra may also be viewed as a residual symmetry of a boundary conformal invariance living on $(1+2)$ space time manifolds. The RdTS supersymmetry we have described is a special FSS because it is related to a deformation of the space-time boundary conformal invariance on AdS_3.

The explicit analysis of this paper has been made plausible due to the particular properties of the AdS_3 geometry:

(a) the AdS_3 manifold carries naturally a $so(1,2)$ affine invariance which has various realization ways;
(b) the Wakimoto representation of the $SO(1,2)$ affine symmetry which on one hand relates its zero mode to the projective symmetry of a BCFT on AdS_3 and on the other hand links the L_- and $\bar{L}_-$ to the translation operators on ∂AdS_3;
(c) the correspondence between WS and space-time symmetries which plays a crucial role in analysing the various kinds of symmetries living on ∂AdS_3.

Finally, we would like to note that this study might find a natural application in FQH systems formulated as an effective Chern–Simon gauge theory. In this approach, the physics in the bulk is roughly speaking described by a $(1+2)$-dimensional $U(1)^n$ gauge model, while the edge excitations of FQH liquids are described by a boundary conformal field theory. We plan to extend the results of this paper to the case of FQH droplets in a future work. Preliminary results in this direction were given in [32].

Acknowledgements. The authors would like to thank Dr. Rausch de Traubenberg for discussions, suggestions and for reading the manuscript. EHS would like to thank E.M. Sahraoui for earlier collaboration on strings on Anti-de Sitter space and A. Belhaj for discussions. This research work is supported by the program PARS Physique 27 under contract 372-98 CNR.

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